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# Auxiliary matrices on both sides of the equator

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**Abstract**

The spectra of previously constructed auxiliary matrices for the six-vertex model at roots of unity are investigated for spin-chains of even and odd length. The two cases show remarkable differences. In particular, it is shown that for even roots of unity and an odd number of sites the eigenvalues contain two linear independent solutions to Baxter's $TQ$-equation corresponding to the Bethe ansatz equations above and below the equator. In contrast, one finds for even spin-chains only one linear independent solution and complete strings. The other main result is the proof of a previous conjecture on the degeneracies of the six-vertex model at roots of unity. The proof rests on the derivation of a functional equation for the auxiliary matrices which is closely related to a functional equation for the eight-vertex model conjectured by Fabricius and McCoy.

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# 1 Introduction

Baxter's $TQ$-equation [1] is one of the corner-stones of integrable systems and has been discussed in a variety of contexts. While it originated from the Bethe ansatz computations for the six-vertex model, it provides conceptually a more general framework to solve the transfer matrix eigenvalue problem of integrable vertex models and paved the way for the solution of the eight-vertex model [2, 3]. (See also [4, 5] for recent developments.)

In the $TQ$-equation the $T$ symbolizes the transfer matrix of the integrable model at hand and $Q$ is called the auxiliary matrix in terms of which the transfer matrix can be expressed. To be concrete, consider the six-vertex model on a square-lattice with periodic boundary conditions. Then the $TQ$-equation is a second order linear difference equation of the form

$$T(z)Q(z) = \phi(zq^{-2})Q(zq^2) + \phi(z)Q(zq^{-2}) \tag{1}$$

with $z \in \mathbb{C}$ being the spectral variable and $q$ the coupling (or crossing) parameter of the model. The coefficients $\phi$ are given by some known scalar function, which in our convention specified below will be $\phi(z) = (zq^2 - 1)^M$. Here $M$ is the number of lattice columns. This equation is often treated on different levels. Some authors interpret it merely on the level of complex-valued functions, i.e. for the eigenvalues of the respective matrices only, not addressing the usually harder problem of the explicit construction of the matrix $Q$. Solving this construction problem, however, allows one to address the problem of determining the existence and number of possible solutions to the $TQ$-equation. This is particularly important for those cases which one often finds to be explicitly (or implicitly) excluded from the discussion, for instance when the parameter $q$ is a root of unity or when the length of the spin-chain associated with the vertex model is odd.

For these "special" cases the relation between the spectrum of explicitly constructed $Q$-matrices in [6, 7, 8, 9] and the solutions of the $TQ$-equation will be our primary interest in this article. For our discussion it will be important to distinguish between the $TQ$-equation as a functional relation and an operator equation.

## 1.1 The $TQ$-equation in terms of functions.

In [10] Krichever et al pointed out that Baxter's $TQ$-equation naturally appears as an auxiliary linear problem in the context of the discretized Liouville equation, see Section 4 therein. For the six-vertex model the analogue of this equation arises from the fusion hierarchy [11, 12],

$$T^{(n)}(zq^2)T^{(n)}(z) - T^{(n+1)}(z)T^{(n-1)}(zq^2) = \phi(z)\phi(zq^{2n}) \ . \tag{2}$$

Here the $T^{(n)}$'s should be thought of as the eigenvalues of the six-vertex fusion matrix of spin $n/2$ with $T^{(2)} = T$ being the transfer matrix and $T^{(1)} = \phi$ the quantum determinant [13]. We will give the definition of the six-vertex fusion hierarchy below. The general exposition given in [10] (which also addresses the case of higher rank and elliptic functions) concludes that there will be in general two-linear independent solutions, say $\mathcal{Q}^{\pm}$, to the functional equation (1) which satisfy the Wronskian

$$\mathcal{Q}^+(zq^2)\mathcal{Q}^-(z) - \mathcal{Q}^+(z)\mathcal{Q}^-(zq^2) = \phi(z) \ . \tag{3}$$

From this Wronskian two types of "complementary" Bethe ansatz equations can be deduced [10]. Notice that after equation (4.27) in [10] the restriction to elliptic polynomials $\phi$ of even degree is made which corresponds to the case of spin-chains of even length. While the focus in [10] is mainly on the elliptic case, one wonders about the trigonometric limit and the implications for the six-vertex model.

Pronko and Stroganov investigated this question of two potential linear independent solutions of the six-vertex $TQ$-equation in [14]. Starting with the XXX spin-chain their discussion is generalized to the XXZ spin-chain excluding the root of unity case. They identify for spin-chains with an odd number of sites $M$ one solution, say $\mathcal{Q}^+$, as the familiar solution from the Bethe ansatz, while the second solution $\mathcal{Q}^-$ is associated with solving the Bethe ansatz on the "wrong side of the equator" meaning that it incorporates $M-n$ Bethe roots. Here $0 \leq n \leq M/2$ is the number of down-spins in the corresponding eigenstate of the transfer matrix or spin-chain Hamiltonian. For spin-chains with an even number of sites, however, there appears to be only one solution in the case of periodic boundary conditions. Baxter addresses this phenomenon on the basis of the coordinate Bethe ansatz and numerical computations in [15] stating that the missing Bethe roots for the second solution $\mathcal{Q}^-$ have gone off to infinity. (See also the third paragraph after equation (4.43) in [10].) When a nonzero horizontal electric field is applied, numerical computations show both linear independent solutions exist for $M$ even and odd [15].

The difference between spin-chains of odd and even length has been further underlined in [16]. For the special root of unity case $q^3 = 1$ and spin-chains with an odd number of sites the two independent solutions of the six-vertex $TQ$-equation have been explicitly constructed [16] starting from a conjecture on the particularly simple form of the groundstate [17, 18, 19].

Our discussion in this article will show that the solutions obtained at $q^3 = 1$ in [16] are in fact the *only* linear independent solutions in the spin $S^z = \pm 1/2$ sector which possess the expected number of $(M-1)/2$ and $(M+1)/2$ Bethe roots, respectively. This is due to the large degeneracies in the spectrum of the transfer matrix connected with the loop algebra symmetry of the six-vertex model at roots of unity [20]. A similar reduction in the number of solutions takes place at higher roots of unity as well. In other words, the number and nature of solutions to the $TQ$-equation (1) does also crucially depend on the value of the parameter $q$ and not only on the length of the spin-chain.

The various cases outlined and the necessary distinctions one has to make clearly show the importance of the explicit construction of auxiliary matrices in order to obtain better control over the different scenarios. However, Baxter's construction of auxiliary matrices for the six and eight-vertex model as presented in his book [21] applies only to spin-chains of even length (see the comment after equation (9.8.16)) and also at roots of unity can only be extended to a particular subset of cases. In contrast, the auxiliary matrices constructed in [7] for roots of unity $q^N = 1$ and the ones in [6] for the case of "generic" $q$ do not have such limitations and apply for even as well as odd length of the chain. They are also of a simpler algebraic form. In this work we will explicitly relate those auxiliary matrices' spectrum to the two linear independent solutions of the $TQ$-equation by extending the discussion of [8] from the even to the odd case.

## 1.2 The $TQ$-equation in terms of operators.

Starting with the papers [22, 23] on the Liouville model and subsequent papers on the six-vertex model [24, 6, 7] a new approach to construct auxiliary matrices has been developed which relies on representation theory. In this method one first solves the Yang-Baxter equation to obtain a matrix which commutes with the transfer matrix and afterwards derives the $TQ$-equation by investigating the decomposition of certain tensor products of representations. The novel feature [6, 7] in this context is the appearance of additional free parameters, collectively called $p$, in the auxiliary matrix, i.e. $Q = Q(z;p)$. These free parameters shift in the operator solution of the $TQ$-equation [6, 7],

$$T(z)Q(z;p) = \phi(zq^{-2})Q(zq^2;p') + \phi(z)Q(zq^{-2};p'') \ . \tag{4}$$

Thus, in this construction method one has to consider a generalized version of Baxter's $TQ$-equation at the level of operators. The free parameters are, for instance, necessary to break spin-reversal symmetry or to lift the degeneracies of the transfer matrix at roots of unity [7]. They also contain information on the analytic structure of the eigenvalues of the auxiliary matrices [8, 9].

For the case when $q$ is not a root of unity and twisted boundary conditions the spectra of of the auxiliary matrices constructed in [6] have been computed using the algebraic Bethe ansatz [9]. It was found that the eigenvalues decompose into two parts which are related by spin-reversal [9],

$$Q(z;p) = Q^+(z;p)Q^-(z;p) \ . \tag{5}$$

For special choices of the parameters $p$ the functions $Q^\pm$ are the eigenvalues of the lattice analogues of the $Q$-operators constructed by Bazhanov et al for the Liouville model [22, 23]. For twisted boundary conditions these two parts of the eigenvalue (5) can indeed be identified (up to some normalization constants) with the two linear independent solutions $\mathcal{Q}^\pm$ of the $TQ$-equation discussed in the previous section, see [9] and the conclusions of this article. However, in general this is not true, since the two linear independent solutions $\mathcal{Q}^\pm$ might not always exist, as for example in the case of periodic boundary conditions and spin-chains of even length. Notice also that Baxter's $Q$-operator for even spin-chains corresponds to only one of these eigenvalue parts, say $Q^+$. This should make clear that the $Q$-operators in [6, 7] not only differ in the construction procedure but are quite different objects from Baxter's $Q$ discussed in [21].

## 1.3 Outline and results of this article

The outline of the article is as follows:

**Section 2.** We introduce the fusion hierarchy of the six-vertex model and fix our conventions. The fusion matrices are defined such that they are polynomials of maximal degree $M$ in the spectral variable $z$. Here $M$ is the length of the associated XXZ spin-chain.

**Section 3.** We recall the definition of a particular subset of the auxiliary matrices at roots of unity constructed in [7] and discuss the general form of their spectra defining the subparts $Q^\pm$ of the eigenvalue decomposition (5). Preliminary results have

already been obtained in [8] for the case of spin-chains with $M$ even. Additional results for $M$ even and odd are contained in [9]. In this article we will complete our investigation of the spectra by extending the discussion of [8] to spin-chains with an odd number of sites.

**Section 4.** The eigenvalues of the auxiliary matrices are in general polynomials in the spectral variable. In this section we determine when they have maximal degree and when they vanish at the origin. Both facts are related to the absence or occurrence of infinite Bethe roots.

**Section 5.** We review from [8] the discussion of the $TQ$-equation in terms of the eigenvalues of the auxiliary matrices. In particular, we discuss the transformation under spin-reversal and show how the second linear independent solution to the $TQ$-equation arises.

**Section 6.** One of the main results of this article is the derivation of a functional equation, for $M$ even and odd, that proves a previously formulated conjecture in [8], see equations (18) and (19) therein. This result will enable us to determine the level of degeneracy of the eigenvalues of the transfer matrix at roots of unity and relate the two parts $Q^{\pm}$ of the eigenvalues via an inversion formula. Moreover, this result can be considered as a six-vertex analogue of the eight-vertex functional equation conjectured by Fabricius and McCoy in [4, 5]. We will comment on this in the conclusions.

**Section 7.** The results for the case of odd spin-chains will show that for even roots of unity and periodic boundary conditions we obtain both linear independent solutions $\mathcal{Q}^{\pm}$ of the $TQ$-equation from the auxiliary matrices in [7]. This is a constructive existence proof for these solutions. For odd roots of unity we will see that the eigenstates of the transfer matrix associated with $\mathcal{Q}^{\pm}$ (in the case that both solutions exist) correspond to zero eigenvalues of the auxiliary matrices. We in particular make contact with Stroganov's solutions for $N = 3$ [16] and show that they are the only ones with the expected number of Bethe roots (see also the appendix).

**Section 8.** We state our conclusions and relate our results to the case when $q$ is not a root of unity considered in [9] and the recent developments in the eight-vertex model [4, 5].

## 2 The six-vertex fusion hierarchy

We introduce the six-vertex model from a representation theoretic point of view. Denote by $\pi_z^{(n)} : U_q(\widetilde{sl}_2) \to \operatorname{End}\mathbb{C}^{n+1}$ the spin $n/2$ evaluation representation of the quantum loop algebra, i.e.

$$\begin{aligned}
\pi_z^{(n)}(e_1)\,|m\rangle &= [n-m+1]_q\,|m-1\rangle\,, \quad \pi_z^{(n)}(f_0) = z^{-1}\pi_z^{(n)}(e_1),\\
\pi_z^{(n)}(f_1)\,|m\rangle &= [m+1]_q\,|m+1\rangle\,, \quad \pi_z^{(n)}(e_0) = z\pi_z^{(n)}(f_1),\\
\pi_z^{(n)}(q^{h_1})\,|m\rangle &= q^{n-2m}\,|m\rangle\,, \quad \pi_z^{(n)}(q^{h_0}) = \pi_z^{(n)}(q^{-h_1}),
\end{aligned} \tag{6}$$

with $m = 0, 1, ..., n$. Define the fusion matrix of degree $n+1$ by setting

$$T^{(n+1)}(zq^{-n-1}) = \operatorname*{Tr}_0 L^{(n+1)}_{0M}(zq^{n+1}) \cdots L^{(n+1)}_{01}(zq^{-n-1}) \tag{7}$$

where $L^{(n+1)}$ is the intertwiner with respect to the tensor product $\pi^{(n)}_w \otimes \pi^{(1)}_1$,

$$\begin{aligned}
\langle 0| L^{(n+1)}(w) |0\rangle &= wq\, \pi^{(n)}(q^{h_1/2}) - \pi^{(n)}(q^{-h_1/2}), \\
\langle 0| L^{(n+1)}(w) |1\rangle &= wq\, (q - q^{-1})\pi^{(n)}(q^{h_1/2})\pi^{(n)}(f_1), \\
\langle 1| L^{(n+1)}(w) |0\rangle &= (q - q^{-1})\pi^{(n)}(e_1)\pi^{(n)}(q^{-h_1/2}), \\
\langle 1| L^{(n+1)}(w) |1\rangle &= wq\, \pi^{(n)}(q^{-h_1/2}) - \pi^{(n)}(q^{h_1/2}) \; .
\end{aligned} \tag{8}$$

Here the scalar products are taken in the second factor of the tensor product, i.e. the spin 1/2 representation. The fusion matrices satisfy the functional equation [11]

$$T^{(n)}(z)T^{(2)}(zq^{-2}) = \left(zq^2 - 1\right)^M T^{(n+1)}(zq^{-2}) + (z-1)^M\, T^{(n-1)}(zq^2) \; . \tag{9}$$

The fusion hierarchy contains two special elements from which all others can be successively generated, namely the six-vertex transfer matrix* $T$ and the quantum determinant $T^{(1)}$ [13] which are obtained via the identification

$$T^{(2)}(zq^{-2}) \equiv T(z) \qquad \text{and} \qquad T^{(1)}(z) \equiv (zq^2 - 1)^M \; . \tag{10}$$

In this manner the above functional equation may also serve as a defining relation for the fusion matrices. An alternative form of the fusion hierarchy is the one given in the introduction, see (2). Both versions are equivalent. From the transfer matrix $T$ we obtain as logarithmic derivative the $XXZ$ spin-chain Hamiltonian,

$$H_{\mathrm{XXZ}} = -\left(q - q^{-1}\right)\; z\frac{d}{dz} \ln \left.\frac{T(z)}{(zq^2-1)^M}\right|_{z=1} \tag{11}$$

$$= -\frac{1}{2}\sum_{m=1}^{M}\left\{ \sigma^x_m \sigma^x_{m+1} + \sigma^y_m \sigma^y_{m+1} + \frac{q+q^{-1}}{2}\left(\sigma^z_m \sigma^z_{m+1} - 1\right)\right\} \; . \tag{12}$$

The well-known symmetries of the model are expressed in terms of the following commutators

$$[T^{(m)}(z), T^{(n)}(w)] = [T^{(n)}(z), S^z] = [T^{(n)}(z), \mathfrak{R}] = [T^{(n)}(z), \mathfrak{S}] = 0 \; , \tag{13}$$

where the respective operators are defined as

$$S^z = \frac{1}{2}\sum_{m=1}^{M}\sigma^z_m, \quad \mathfrak{R} = \sigma^x \otimes \cdots \otimes \sigma^x = \prod_{m=1}^{M}\sigma^x_m, \quad \mathfrak{S} = \sigma^z \otimes \cdots \otimes \sigma^z = \prod_{m=1}^{M}\sigma^z_m \; . \tag{14}$$

These symmetries hold for spin-chains of even as well as odd length.

*Note that our definition of the six-vertex transfer matrix differs from the one in [7, 8, 9] by an overall factor,

$$T(z) \to q^{\frac{M}{2}}\, T(z)/(zq^2 - 1)^M \; .$$

## 3 Auxiliary matrices at roots of unity

In a series of papers [7, 8, 9] auxiliary matrices for the six-vertex model at roots of unity have been constructed. The construction procedure and the difference with Baxter's method have been discussed in [7] and we refer the reader to this work for details. In order to keep this paper self-contained we briefly give the definition of a special subvariety of the set of auxiliary matrices constructed in [7].

### 3.1 Definition

Suppose $q$ is a primitive root of unity of order $N$ and set $N'=N$ if the order is odd and $N'=N/2$ if it is even. Define the following $N'$-dimensional nilpotent evaluation representation $\pi_w^\mu$ of the quantum loop algebra $U_q(\widetilde{sl}_2)$ [25, 26],

$$\begin{aligned}\pi^\mu(q^{h_1})\,|n\rangle &= \mu^{-1}q^{-2n-1}\,|n\rangle\,,\quad \pi^\mu(f_1)\,|n\rangle=|n+1\rangle\;,\quad \pi^\mu(f_1)\,\big|N'-1\big\rangle=0\;,\\ \pi^\mu(e_1)\,|n\rangle &= \frac{\mu+\mu^{-1}-\mu q^{2n}-\mu^{-1}q^{-2n}}{(q-q^{-1})^2}\;|n-1\rangle\,,\end{aligned}\tag{15}$$

and

$$\pi_w^\mu(q^{h_0})=\pi^\mu(q^{-h_1}),\quad \pi_w^\mu(f_0)=w^{-1}\pi^\mu(e_1),\quad \pi_w^\mu(e_0)=w\;\pi^\mu(f_1),\quad w,\mu\in\mathbb{C}^\times\;.\tag{16}$$

Let the matrix

$$L^\mu=\begin{pmatrix}\alpha_\mu & \beta_\mu\\ \gamma_\mu & \delta_\mu\end{pmatrix}=\alpha_\mu\otimes\sigma^+\sigma^-+\beta_\mu\otimes\sigma^++\gamma_\mu\otimes\sigma^-+\delta_\mu\otimes\sigma^-\sigma^+\;.\tag{17}$$

be the intertwiner of the tensor product $\pi_w^\mu\otimes\pi_{z=1}^{(1)}$ of evaluation representations, explicitly

$$\begin{aligned}\alpha_\mu(w) &= wq\,\pi^\mu(q^{h_1/2})-\pi^\mu(q^{-h_1/2}),\quad \beta_\mu(w)=wq(q-q^{-1})\pi^\mu(q^{h_1/2})\pi^\mu(f_1),\\ \gamma_\mu &= \left(q-q^{-1}\right)\pi^\mu(e_1)\pi^\mu(q^{-h_1/2}),\quad \delta_\mu(w)=wq\,\pi^\mu(q^{-h_1/2})-\pi^\mu(q^{h_1/2})\;.\end{aligned}\tag{18}$$

Define the auxiliary matrix in terms of these matrices as the trace of the following operator product,

$$Q_\mu(z)=\operatorname*{Tr}_0 L^\mu_{0M}(z/\mu)L^\mu_{0M-1}(z/\mu)\cdots L^\mu_{01}(z/\mu)\;.\tag{19}$$

This matrix commutes by construction with the fusion matrices,

$$[Q_\mu(w),T^{(n)}(z)]=0,\tag{20}$$

and preserves two of the symmetries (13) [7, 8],

$$[Q_\mu(z),S^z]=[Q_\mu(z),\mathfrak{S}]=0\;.\tag{21}$$

Spin-reversal symmetry on the other hand is broken [7, 8],

$$\mathfrak{R}Q_\mu(z,q)\mathfrak{R}=Q_{\mu^{-1}}(z\mu^{-2},q)=Q_{\mu^{-1}}(zq^2\mu^{-2},q^{-1})^t=(-zq/\mu)^MQ_\mu(z^{-1}q^{-2}\mu^2)^t\;.\tag{22}$$

These relations hold for all $M$ and allow one to determine the conjugate transpose of the auxiliary matrix [8],

$$Q_\mu(z,q)^* = Q_{\bar\mu}(\bar z, q^{-1})^t = Q_{\bar\mu}(\bar z q^{-2}, q)\ . \tag{23}$$

In addition, one derives from the following non-split exact sequence of evaluation representations $\pi^\mu_w$ [7]

$$0 \to \pi^{\mu q}_{w'} \hookrightarrow \pi^\mu_w \otimes \pi^{(1)}_z \to \pi^{\mu q^{-1}}_{w''} \to 0, \quad w = w'q^{-1} = w''q = z/\mu \tag{24}$$

the $TQ$-equation

$$T(z)Q_\mu(z) = (z-1)^M\ Q_{\mu q}(zq^2) + (zq^2-1)^M\ Q_{\mu q^{-1}}(zq^{-2})\ . \tag{25}$$

The proof can be found in [7], here we will only review parts of the calculation of the spectrum of the auxiliary matrices given in [8] and extend the results therein to spin-chains of odd length.

### 3.2 The general form of the spectrum

The starting point is the same as in [8]: provided that the commutation relation

$$[Q_\mu(z), Q_\nu(w)] = 0, \qquad \mu,\nu,z,w \in \mathbb{C} \tag{26}$$

holds, the eigenvectors of $Q_\mu(z)$ are independent of the parameter $\mu$ as well as the spectral variable $z$. In order to prove (26) one has to explicitly construct the corresponding intertwiners of the tensor products $\pi^\mu_w \otimes \pi^\nu_1$ for all $N' \in \mathbb{N}$. As pointed out in [8] the necessary conditions for these intertwiners to exist are satisfied for all roots of unity. An explicit construction has been carried out for $N = 3$ [8] and $N = 6$. Numerical checks have been performed for $N' = 4, 5, 7$. We shall take this as sufficient evidence for (26) to hold true.

There are two important implications of (26). The first is that the auxiliary matrices are normal and hence diagonalizable, see (23). The second consequence is that the eigenvalues of $Q_\mu$ must be polynomial in the spectral variable $z$. Their most general form is therefore given by†

$$\begin{aligned} Q_\mu(z) &= \mathcal{N}_\mu\, z^{n_\infty} P_B(z) P_\mu(z) P_S(z^{N'}, \mu^{2N'}) \\ &= \mathcal{N}_\mu\, z^{n_\infty} \prod_{i=1}^{n_+}(1-z/z_i) \prod_{i=1}^{n_-}(1-z/w_i(\mu)) \prod_{i=1}^{n_S}(1-z^{N'}/a_i(\mu))\ . \end{aligned} \tag{27}$$

Note that we slightly differ in the notation from [8] and have redefined the normalization constant $\mathcal{N}_\mu$ by setting $P_B(0) = P_\mu(0) = P_S(0) = 1$. Besides these minor differences our definition of the various polynomials entering the eigenvalues is the same as in [8].

- The monomial factor in front of the eigenvalue is related to the occurrence of vanishing and infinite Bethe roots when the root of unity limit is taken in the deformation parameter $q$.

†Throughout this paper we will denote eigenvalues and operators by the same symbol.

- The second factor $P_B$ contains only roots $z_i$ which do not depend on the free parameter $\mu$ and which will be identified with the finite Bethe roots at roots of unity. Moreover, we exclude from this set complete or exact strings, i.e. for every $z_i$ there is at least one integer $0 < \ell < N'$ such that $z_i q^{2\ell}$ is not a root of $P_B$.

- The third factor contains roots which do depend on the parameter $\mu$. For even chains, $M \in 2\mathbb{N}$, this factor was identified in [8] with the rescaled polynomial $P_B$,

$$P_\mu(z) = P_B(z\mu^{-2}) \ .$$

  Here we will find that this relation ceases to be valid for even roots of unity when $M \in 2\mathbb{N}+1$. Again we exclude the possibility that the roots of $P_\mu$ occur in strings.

- Finally, the last factor $P_S$ contains all roots which occur in strings. The zeroes $a_i$ may or may not depend on the parameter $\mu$. Because all roots are sitting in a string the polynomial depends on $z^{N'}$ rather than $z$. Note that we allow for the possibility $n_S = 0$.

For later purposes let us decompose the eigenvalues of the auxiliary matrices into two parts similar as it has been done in [23, 9] for $q$ "generic". Namely, we set

$$Q^+(z) = P_B(z) = \prod_{i=1}^{n_+}(1 - z/z_i) \tag{28}$$

and secondly,

$$Q^-(z) = \lim_{\mu \to q^{-N'}} \mathcal{N}_\mu \, z^{n_\infty} P_\mu(z) P_S(z^{N'}, \mu^{2N'}) \ . \tag{29}$$

This decomposition may seem arbitrary at the moment but it will become clear in our line of argument. Notice that we have eliminated the dependence of the auxiliary matrix on the free parameter $\mu$ in (29). Moreover, it is worth stressing that $Q^\pm$ are not always both identical with the two linear independent solutions $\mathcal{Q}^\pm$ of the $TQ$-equation (the latter might even not exist),

$$T(z) \equiv T^{(2)}(zq^{-2}) = (zq^2 - 1)^M q^{\mp S^z} \frac{\mathcal{Q}^\pm(zq^{-2})}{\mathcal{Q}^\pm(z)} + (z-1)^M q^{\pm S^z} \frac{\mathcal{Q}^\pm(zq^2)}{\mathcal{Q}^\pm(z)} \tag{30}$$

which satisfy the Wronskian

$$q^{S^z} \mathcal{Q}^+(zq^2)\mathcal{Q}^-(z) - q^{-S^z} \mathcal{Q}^+(z)\mathcal{Q}^-(zq^2) = (q^{S^z} - q^{-S^z})(1 - zq^2)^M \ . \tag{31}$$

The additional phase factors $q^{\pm S^z}$ in comparison with the equation (3) of [10] discussed in the introduction are due to different conventions. We shall set $\mathcal{Q}^+(0) = \mathcal{Q}^-(0) = 1$ and $\deg \mathcal{Q}^+ = M - \deg \mathcal{Q}^- = M/2 - S^z$. While $Q^+$ is by definition the solution $\mathcal{Q}^+$ above the equator, $Q^- \neq \mathcal{Q}^-$ in general. In particular we have included the normalization constant $\lim_{\mu \to q^{-N'}} \mathcal{N}_\mu$ in the definition of $Q^-$, which in some cases can be zero as we will discuss below.

# 4 The degree of the eigenvalues and "infinite" Bethe roots

We start our discussion with the first factor in the eigenvalue (27), the monomial $z^{n_\infty}$ which is related to the fact that some Bethe roots in the root of unity limit vanish or tend to infinity. Obviously, $n_\infty \neq 0$ if and only if the eigenvalue of the auxiliary matrices vanishes at the origin. Another obvious observation is that by construction of the auxiliary matrices it follows that

$$\deg Q_\mu = n_\infty + n_+ + n_- + n_S N' \leq M \ . \tag{32}$$

As it turns out this upper bound is assumed if and only if we have a vanishing monomial contribution, i.e. $n_\infty = 0$. This can be deduced from the following relation for the auxiliary matrices [7, 8]

$$Q_\mu(z) = (-zq/\mu)^M Q_{\mu^{-1}}(z^{-1}q^{-2})^t \tag{33}$$

which implies that the coefficients in the power series expansion

$$Q_\mu(z) = \sum_{m=0}^{M} Q_\mu^{(m)} z^m \tag{34}$$

are related via

$$Q_\mu^{(m)} = (-\mu)^{-M} q^{-M+2m} \left( Q_{\mu^{-1}}^{(M-m)} \right)^t \ . \tag{35}$$

In particular, setting $m = 0$ in the above identity we see that the eigenvalue of $Q_\mu$ is of degree $M$ whenever $Q_\mu(0) = Q_\mu^{(0)} \neq 0$. At the same time this clearly prevents $n_\infty \neq 0$. The coefficient $Q_\mu^{(0)}$ can be easily calculated by noting that the building blocks $L^\mu(z)$ of the auxiliary matrix are lower triangular matrices at $z = 0$,

$$L^\mu(0) = -\pi^\mu(q^{-h_1/2}) \otimes \sigma^+\sigma^- - \pi^\mu(q^{h_1/2}) \otimes \sigma^-\sigma^+ + (q - q^{-1})\pi^\mu(e_1 q^{-h_1/2}) \otimes \sigma^- \ . \tag{36}$$

Thus, the matrix $Q_\mu(0)$ in quantum space is diagonal with its diagonal elements given by

$$Q_\mu(0) = (-)^M \operatorname*{Tr}_{\pi^\mu} q^{-h_1 S^z} = (-)^M (\mu q)^{S^z} \sum_{\ell=0}^{N'-1} q^{2\ell S^z} \ . \tag{37}$$

For the interpretation of this result we distinguish the following cases:

1. When $q^{N'} = 1$, i.e. for primitive roots of unity of odd order, the degree of the polynomial will only be equal to $M$ in the commensurate sectors $2S^z = 0 \bmod N$. At the same time this means that there are no infinite Bethe roots, that is $n_\infty = 0$. Consequently, we have

$$Q_\mu(0) = \mathcal{N}_\mu = (-)^M N \, q^{S^z} \mu^{S^z}, \qquad q^{N'} = 1, \ 2S^z = 0 \bmod N \ . \tag{38}$$

2. When $q^{N'} = -1$ we have to distinguish between $M$ even and odd. Let $M$ be even then $S^z$ takes integer values only and we obtain $\deg Q_\mu = M$ if and only if $2S^z = 0 \bmod N$. Again we find in these spin-sectors the normalization constant

$$Q_\mu(0) = \mathcal{N}_\mu = N' q^{S^z} \mu^{S^z}, \qquad q^{N'} = -1, \ M \in 2\mathbb{N}, \ 2S^z = 0 \bmod N \ . \tag{39}$$

3. For $q^{N'}=-1$ and $M$ odd, however, the total spin eigenvalue will be constrained to the set $2S^z\in 2\mathbb{Z}+1$, preventing the existence of a monomial factor. That is, in this case we always have $\deg Q_\mu = M$ and $n_\infty=0$. The normalization constant is therefore

$$Q_\mu(0)=\mathcal{N}_\mu=-q^{N'S^z}\mu^{S^z}\ \frac{q^{N'S^z}-q^{-N'S^z}}{q^{S^z}-q^{-S^z}},\qquad q^{N'}=-1,\ M\in 2\mathbb{N}+1\ . \tag{40}$$

# 5 The $TQ$-equation

The most important property of the auxiliary matrix is the solution of the following functional equation with the six-vertex transfer matrix which has been proved to hold for $M$ even and odd [7],

$$T(z)Q_\mu(z)=(z-1)^M\ Q_{\mu q}(zq^2)+(zq^2-1)^M\ Q_{\mu q^{-1}}(zq^{-2})\ . \tag{41}$$

From this functional equation and the fact that $T^{(2)}$ does not depend on the free parameter $\mu$ we infer similar to the case $M$ even considered in [8] that

$$w_i(\mu)=w_i\mu^2\qquad\text{and}\qquad a_i(\mu)=a_i(\mu^{2N'})$$

implying the following form for the eigenvalues of the transfer matrix,

$$T(z)=\frac{\mathcal{N}_{\mu q}}{\mathcal{N}_\mu}\ q^{2n_\infty}\ (z-1)^M\ \frac{Q^+(zq^2)}{Q^+(z)}+\frac{\mathcal{N}_{\mu q^{-1}}}{\mathcal{N}_\mu}\ q^{-2n_\infty}(zq^2-1)^M\ \frac{Q^+(zq^{-2})}{Q^+(z)}\ . \tag{42}$$

Here the ratios of the normalization constants can only depend on $q$ from which we deduce

$$\frac{\mathcal{N}_{\mu q}}{\mathcal{N}_\mu}=\frac{\mathcal{N}_\mu}{\mathcal{N}_{\mu q^{-1}}}\ . \tag{43}$$

The zeroes of the polynomial $Q^+$ are fixed through the "Bethe ansatz" equations,

$$0=(1-z_iq^2)^Mq^{-s}Q^+(z_iq^{-2})+(1-z_i)^Mq^sQ^+(z_iq^2),\qquad i=1,...,n_+, \tag{44}$$

with

$$q^s:=\frac{\mathcal{N}_{\mu q}}{\mathcal{N}_\mu}q^{2n_\infty}=\frac{\mathcal{N}_\mu}{\mathcal{N}_{\mu q^{-1}}}q^{2n_\infty}\ . \tag{45}$$

We will argue below that this phase factor is determined by the total spin of the eigenstate and the number of Bethe roots which tend to zero in the root of unity limit.

## 5.1 Spin reversal and Bethe roots "beyond the equator"

We are now exploring the role of the polynomial factor $P_\mu$. Following the same line of argument as in [8] we act with the spin reversal operator from both sides on the $TQ$-equation employing the transformation law [7, 8]

$$\mathfrak{R}Q_\mu(z)\mathfrak{R}=Q_{\mu^{-1}}(z\mu^{-2})$$

of the auxiliary matrix. Replacing afterwards $\mu \to \mu^{-1}$ we obtain the equation

$$T(z)Q_\mu(z\mu^2) = (z-1)^M \; Q_{\mu q^{-1}}(z\mu^2) + (zq^2-1)^M \; Q_{\mu q}(z\mu^2) \; . \tag{46}$$

Since we employed the spin-reversal operator we refer to this identity as the $TQ$-equation "beyond the equator". The corresponding expression in terms of eigenvalues is deduced to be

$$\begin{aligned} T(z) &= \frac{\mathcal{N}_{\mu q^{-1}}}{\mathcal{N}_\mu} \; (z-1)^M \, \frac{P_{\mu q^{-1}}(z\mu^2)}{P_\mu(z\mu^2)} + \frac{\mathcal{N}_{\mu q}}{\mathcal{N}_\mu} \; (zq^2-1)^M \, \frac{P_{\mu q}(z\mu^2)}{P_\mu(z\mu^2)} \\ &= (z-1)^M \, q^{-s} \, \frac{Q^-(zq^2)}{Q^-(z)} + (zq^2-1)^M q^s \, \frac{Q^-(zq^{-2})}{Q^-(z)} \; . \end{aligned} \tag{47}$$

Notice that in comparison with (42) not only the phase factors $q^s$ have been inverted but that the polynomial $Q^-$ will in general have a different degree than $Q^+$, i.e. $n_- \neq n_+$. From (46) we now obtain the "Bethe ansatz equations beyond the equator",

$$0 = (1-w_i q^2)^M q^s Q^-(w_i q^{-2}) + (1-w_i)^M q^{-s} Q^-(w_i q^2), \quad i = 1, ..., n_- \; . \tag{48}$$

This second solution to the $TQ$-equation, which is related by spin-reversal to $Q^+$, will not always be linear independent as we shall see below. The parameter $s$ entering the phase factors in the eigenvalue expressions of the transfer matrices changes sign, since we already saw for particular cases, see (38), (39), (40), that it is related it to the total spin of the corresponding eigenvectors through the normalization constant. For the general case its value can be determined by making contact with the fusion hierarchy.

Using the functional equation (9) for the fusion matrices presented in the introduction, the above results for the transfer matrix are extended to all fusion matrices via induction. A straightforward calculation yields

$$T^{(n)}(z) = q^{\pm(n+1)s} Q^\pm(z) Q^\pm(zq^{2n}) \sum_{\ell=1}^{n} \frac{q^{\mp 2\ell s}(zq^{2\ell}-1)^M}{Q^\pm(zq^{2\ell})Q^\pm(zq^{2\ell-2})} \; . \tag{49}$$

In the paper [9] it has been argued, using the algebraic Bethe ansatz when $q$ is not a root of unity, that the parameter $s$ can be identified with

$$s = 2n_0 + S^z \bmod N' \; . \tag{50}$$

Here $n_0$ denotes the number of Bethe roots which vanish in the root of unity limit $q^N \to 1$.

# 6 A functional equation relating $Q^+$ and $Q^-$

The final step in the analysis of the spectrum of the auxiliary matrices rests on the following functional equation, which has been proved for $N = 3$ in [8],

$$Q_\mu(z\mu^2 q^2)Q_\nu(z) = (zq^2-1)^M Q_{\mu\nu q}(z\mu^2 q^2) + q^{N'M} Q_{\mu\nu q^{-N'+1}}(z\mu^2 q^2) T^{(N'-1)}(zq^2) \; . \tag{51}$$

This equation is a direct consequence of the following decomposition of the tensor product $\pi^\mu_w \otimes \pi^\nu_1$ of evaluation representations,

$$0 \to \pi^{\mu'}_{w'} \hookrightarrow \pi^\mu_w \otimes \pi^\nu_1 \to \pi^{\mu''}_{w''} \otimes \pi^{(N'-2)}_{z'} \to 0 \tag{52}$$

where

$$w=\mu\nu q^2,\quad \mu'=\mu\nu q,\quad w'=\mu q,\quad \mu''=\mu\nu q^{-N'+1},\quad w''=\mu q^{-N'+1},\quad z'=\nu q^{N'+1}\;. \tag{53}$$

For the moment assume the functional equation (51) to hold, the derivation of (51) and (52) is given in the appendix. Let us insert the explicit form of the eigenvalue into (51). We find

$$\frac{Q_\mu(z\mu^2q^2)Q_\nu(z)}{Q_{\mu\nu q}(z\mu^2q^2)}=$$
$$\frac{\mathcal{N}_\mu\mathcal{N}_\nu}{\mathcal{N}_{\mu\nu q}}\,z^{n_\infty}P_B(z)P_{\mu=1}(zq^2)\prod_{i=1}^{n_S}\frac{(1-z^{N'}\mu^{2N'}/a_i(\mu^{2N'}))(1-z^{N'}/a_i(\nu^{2N'}))}{(1-z^{N'}\mu^{2N'}/a_i(\mu^{2N'}\nu^{2N'})}=$$
$$\frac{\mathcal{N}_{\mu\nu q^{N'+1}}}{\mathcal{N}_{\mu\nu q}}\,q^{N'M}T^{(N'-1)}(zq^2)+(zq^2-1)^M$$

Here we have exploited the fact that the zeroes of the factors $P_\mu$, $P_\nu$ and $P_{\mu\nu q^{N'+1}}$, $P_{\mu\nu q}$ only depend on $\mu^2,\nu^2$ and $\mu^2\nu^2q^2$, respectively. The possible complete string contribution $P_S$ contains the various parameters only to the power $2N'$. Notice that the last line of the above equation is independent of the free parameters $\mu,\nu$ (the ratio of the normalization constants only depends on $q$ as pointed out earlier). This implies that the zeroes $a_i(\mu^{2N'})$ are either independent of $\mu$ altogether or only incorporate it as a multiplicative factor, i.e. one has (exactly as in the case $N=3$ proved in [8]) the alternative

$$a_i(\mu^{2N'})=a_i\qquad\text{or}\qquad a_i(\mu^{2N'})=a_i\;\mu^{2N'}\;. \tag{54}$$

Hence, there are at most $2^{n_S}$ possible eigenvalues of the auxiliary matrix in a degenerate eigenspace of the transfer matrix with fixed $n_\infty$ and $n_\pm$. This proves part of the second conjecture made in [8], see equation (18) with $a_i\equiv(z_i^S)^{N'}$. From the outcome on the zeroes of the complete string contribution one deduces that the factor originating from the $P_S$ polynomials simplifies,

$$\prod_{i=1}^{n_S}\frac{(1-z^{N'}\mu^{2N'}/a_i(\mu^{2N'}))(1-z^{N'}/a_i(\nu^{2N'}))}{(1-z^{N'}\mu^{2N'}/a_i(\mu^{2N'}\nu^{2N'})}=P_S(z^{N'},\mu=1)\;. \tag{55}$$

Hence, we can rewrite the functional equation in terms of $Q^\pm$ as follows,

$$\begin{aligned}\frac{Q_\mu(z\mu^2q^2)Q_\nu(z)}{Q_{\mu\nu q}(z\mu^2q^2)}&=\frac{\mathcal{N}_\mu}{\mathcal{N}_{\mu q^{-N'+1}}}\,q^{-2n_\infty}Q^+(z)Q^-(zq^2)\\&=\frac{\mathcal{N}_{\mu q^{2N'+1}}}{\mathcal{N}_{\mu q^{-N'+1}}}\,q^{N'M}T^{(N'-1)}(zq^2)+(zq^2-1)^M\;.\end{aligned}$$

Here we have set $\nu=q^{-N'}$ in the normalization constants without loss of generality. Invoking now the earlier stated form of the fusion matrices (49) this identity becomes

$$\frac{\mathcal{N}_\mu}{\mathcal{N}_{\mu q^{-N'+1}}}\,q^{-2n_\infty}Q^+(z)Q^-(zq^2)=\frac{\mathcal{N}_{\mu q}}{\mathcal{N}_{\mu q^{-N'+1}}}q^{-s}Q^+(z)Q^-(zq^2)=$$
$$\frac{\mathcal{N}_{\mu q^{2N'+1}}}{\mathcal{N}_{\mu q^{-N'+1}}}\,q^{N'M\pm N's}Q^\pm(z)Q^\pm(zq^2)\sum_{\ell=1}^{N'-1}\frac{q^{\mp2\ell s}(zq^{2\ell+2}-1)^M}{Q^\pm(zq^{2\ell+2})Q^\pm(zq^{2\ell})}+(zq^2-1)^M\;.$$

Setting $\mu \to q^{-N'}$ and solving the last expression for $Q^{\pm}$ we obtain the result

$$Q^{\mp}(z) = q^{\pm(N'+1)s} Q^{\pm}(z) \sum_{\ell=1}^{N'} \frac{q^{-2\ell s}(zq^{2\ell}-1)^M}{Q^{\pm}(zq^{2\ell})Q^{\pm}(zq^{2\ell-2})} \,. \tag{56}$$

Hence, the two solutions of the $TQ$-equation are related to each other by an inversion formula (provided that $Q^- \neq 0$, see the discussion below). This identity is the six-vertex analogue of the functional equation conjectured by Fabricius and McCoy for Baxter's 1972 auxiliary matrix of the eight-vertex model [4, 5]. (We will comment further on this in the conclusion.) Here we have proved this functional equation in the six-vertex limit for all roots of unity and spin-chains of even as well as odd length. Let us investigate the difference between the solutions $Q^{\pm}$ depending on the cases when $M$ is even or odd.

**Decomposition of the eigenvalue for $M$ even.** For spin-chains of even length $M = 2m$ and at $q^{N'} = \pm 1$ the sum in (29), which at first sight appears to be a rational function, simplifies due to the Bethe ansatz equations (44) to a polynomial, i.e. we have for any contour $C$ encircling the point $z_i q^{-2\ell'}$,

$$\frac{1}{2\pi i} \oint\limits_{C(z_i q^{-2\ell'})} \sum_{\ell=1}^{N'} \frac{q^{-2\ell s}(zq^{2\ell}-1)^M}{Q^+(zq^{2\ell})Q^+(zq^{2\ell-2})}\, dz =$$
$$\frac{q^{-2\ell' s}(z_i-1)^M}{Q^+(z_i q^{-2})\prod_{j\neq i}(1-z_i/z_j)} + \frac{q^{-2(\ell'+2)s}(z_i q^2-1)^M}{Q^+(z_i q^2)\prod_{j\neq i}(1-z_i/z_j)} = 0, \qquad 0 < \ell' \leq N' \,.$$

As a consequence we obtain in general the identifications [8]

$$\lim_{\mu \to q^{-N'}} P_\mu(z) = Q^+(z) \tag{57}$$

and

$$\lim_{\mu \to q^{-N'}} \mathcal{N}_\mu \, z^{n_\infty} P_S(z^{N'}, \mu^{2N'}) = q^{(N'+1)s} \sum_{\ell=1}^{N'} \frac{q^{-2\ell s}(zq^{2\ell}-1)^M}{Q^+(zq^{2\ell})Q^+(zq^{2\ell-2})} \,. \tag{58}$$

We therefore conclude that $Q^- \neq \mathcal{Q}^-$ in this case. In fact, one finds numerically that the solution $\mathcal{Q}^-$ with $M/2 + S^z$ Bethe roots does not exist. The other important conclusion is that the result (58) together with (54) enables us to read off the degeneracy of the transfer matrix eigenvalue corresponding to $Q^+$. According to (54) each zero $a_i$ in the complete string contribution $P_S$ is either independent of $\mu$ or is multiplied by a factor $\mu^{2N'}$ showing that there are $2^{n_S}$ possible eigenvalues of the auxiliary matrices each corresponding to a vector in the degenerate eigenspace of the transfer matrix. This is in agreement with the observation [20] that only spin-1/2 representations occur in the tensor products describing the finite-dimensional representations of the loop algebra. Obviously, if $n_S = 0$ the eigenvalue of the transfer matrix is non-degenerate up to spin-reversal symmetry.

**Decomposition of the eigenvalue for $M$ odd.** For $M = 2m + 1$ odd the above simplification of the eigenvalue in general also holds true when $N$ odd with the possible exception that the part (58) of the eigenvalue completely vanishes, we will discuss this case below. For $N$ even, however, the polynomials $P_\mu$ and $P_B = Q^+$ differ. Since we have now $s \in \frac{1}{2}\mathbb{Z}$ and $q^{N'} = -1$ the rational function (58) has poles at $z = z_i$ with non-vanishing residue,

$$\frac{1}{2\pi i}\oint_{C(z_i)} \sum_{\ell=1}^{N'} \frac{q^{-2\ell s}(1 - zq^{2\ell})^M}{Q^+(zq^{2\ell})Q^+(zq^{2\ell-2})}\, dz =$$

$$\frac{q^{-2s}(z_i q^2 - 1)^M}{Q^+(z_i q^2)\prod_{j\neq i}(1 - z_i/z_j)} + \frac{q^{-2N's}(z_i - 1)^M}{Q^+(z_i q^{-2})\prod_{j\neq i}(1 - z_i/z_j)} \neq 0$$

Therefore, the factor $Q^+(z)$ in front of the sum is needed to cancel these poles and the above factorization (57), (58) does not take place. As a consequence complete strings are absent and there are no additional degeneracies other than spin-reversal symmetry. As argued earlier in Section 4 infinite Bethe roots are absent as well and we must have,

$$Q^-(z) = \lim_{\mu \to q^{-N'}} \mathcal{N}_\mu\ P_\mu(z) = q^{(N'+1)S^z} Q^+(z) \sum_{\ell=1}^{N'} \frac{q^{-2\ell S^z}(zq^{2\ell} - 1)^M}{Q^+(zq^{2\ell})Q^+(zq^{2\ell-2})} \tag{59}$$

with the degrees of the polynomials $Q^\pm$ obeying

$$\deg Q^- = M - \deg Q^+ = \frac{M}{2} + S^z\ . \tag{60}$$

Thus, we obtain a very different picture depending on the length of the spin-chain being odd or even.

**The quantum Wronskian.** The difference between the two situations of even and odd spin-chains is highlighted further by introducing the analogue of (31) for the two different parts of the auxiliary matrix eigenvalues. This corresponds to the "quantum Wronskian" in [23]. First note that using (56) we easily obtain

$$q^{ns}Q^+(zq^{2n})Q^-(z) - q^{-ns}Q^+(z)Q^-(zq^{2n}) = (q^{N's} - q^{-N's})T^{(n)}(z)\ . \tag{61}$$

Upon specializing to $n = 1$ this relation simplifies to

$$q^{s}Q^+(zq^{2})Q^-(z) - q^{-s}Q^+(z)Q^-(zq^{2}) = (q^{N's} - q^{-N's})(zq^2 - 1)^M\ . \tag{62}$$

Notice that the right hand of the above equation always vanishes except for odd spin-chains and even roots of unity. This signals the linear dependence between $Q^\pm$ for $M$ even and $q^{N'} = 1$, $M$ odd as described above, compare with (57) and (58).

For $M$ odd and $q^{N'} = -1$ the quantum Wronskian is non-zero and we can identify

$$Q^+ = \mathcal{Q}^+ \qquad \text{and} \qquad Q^- = \mathcal{N}_{\mu=q^{-N'}}\mathcal{Q}^- = -\frac{q^{N'S^z} - q^{-N'S^z}}{q^{S^z} - q^{-S^z}}\ \mathcal{Q}^-\ . \tag{63}$$

Thus, via an explicit construction of diagonalizable $Q$-operators we have shown existence of the solutions above and below the equator. Notice that the Wronskian implies the Bethe ansatz equations (44). Namely, we have for each zero $z_i$ of $Q^+$ that

$$q^{S^z}\frac{Q^+(z_iq^2)}{(z_iq^2-1)^M}=\frac{q^{N'S^z}-q^{-N'S^z}}{Q^-(z_i)}=-q^{S^z}\frac{Q^+(z_iq^{-2})}{(z_i-1)^M}\ . \tag{64}$$

An analogous relation holds for the zeroes $w_i$ of $Q^-$ leading to the Bethe ansatz equations beyond the equator (48). Note that (31), respectively (62), contains more information than each copy of the Bethe ansatz equations by itself, as it relates the zeroes $z_i$ and $w_i$ through the following sum rules for each $0\leq m\leq M$,

$$\binom{M}{m}=\sum_{k+\ell=m}\frac{q^{S^z-2\ell}e_k^+e_\ell^- - q^{-S^z-2k}e_k^+e_\ell^-}{q^{S^z}-q^{-S^z}}\ . \tag{65}$$

Here we have in light of (63) identified the zeroes of $\mathcal{Q}^\pm$ with $\{z_i\}$ and $\{w_i\}$ and introduced the elementary symmetric polynomials

$$e_k^+=e_k(z_1^{-1},...,z_{n_+}^{-1})\qquad\text{and}\qquad e_k^-=e_k(w_1^{-1},...,w_{M-n_+}^{-1})\ . \tag{66}$$

Numerically it is by far more feasible to solve this set of $M$ equations, which is *quadratic* in the $M$ variables $\{e_k^+\}\cup\{e_k^-\}$, rather than the original $n_+=M/2-S^z$ Bethe ansatz equations (44) which are of order $M$ in the $n_+$ variables $\{z_i\}$. We verified for $N'=3,5$ up to spin-chains of length $M=11$ that the number of solutions of the equations (65) matches the dimension of the respective eigenspaces of the transfer matrix.

# 7 Zero eigenvalues at $N$ and $M$ odd

An additional aspect in which the cases of even and odd spin-chains differ is the occurrence of zero eigenvalues of the auxiliary matrices. That the auxiliary matrices can have indeed a non-trivial kernel for odd $M$ has already been remarked upon in [7] where it was noted for the simple case of the $M=3$ spin-chain. As we will see it is closely connected with the inversion formula (56) which follows from the functional equation (51) and the two independent solutions $\mathcal{Q}^\pm$ of the $TQ$-equation.

**Eigenstates with a maximum number of Bethe roots.** Suppose $q$ is an odd root of unity then, as discussed above, the rational function (58) becomes a polynomial in $z$ and $P_\mu(z)=Q^+(z\mu^{-2})$. The only exception to this scenario is the case when the corresponding eigenstate of the transfer matrix is a singlet. According to our previous discussion, we therefore must have $2^{n_S}=1$, i.e. the corresponding eigenvalue of the auxiliary matrix cannot contain complete strings. In the absence of infinite Bethe roots we now argue that this implies (58) vanishes. This can be deduced in several ways. Suppose the two linear independent solutions $\mathcal{Q}^\pm$ to the $TQ$-equation exist. Then they have to obey the quantum Wronskian (31). Solving the latter for $\mathcal{Q}^-$ we obtain

$$\mathcal{Q}^-(z)=q^{-2S^z}\frac{\mathcal{Q}^+(z)\mathcal{Q}^-(zq^2)}{\mathcal{Q}^+(zq^2)}+(1-q^{-2S^z})\frac{(1-zq^2)^M}{\mathcal{Q}^+(zq^2)}\ . \tag{67}$$

Iteration of this formula yields after $N$-steps

$$\mathcal{Q}^-(z)=q^{-2NS^z}\frac{\mathcal{Q}^+(z)\mathcal{Q}^-(zq^{2N})}{\mathcal{Q}^+(zq^{2N})}+(q^{2S^z}-1)\mathcal{Q}^+(z)\sum_{\ell=1}^{N}\frac{q^{-2\ell S^z}(1-zq^{2\ell})^M}{\mathcal{Q}^+(zq^{2\ell})\mathcal{Q}^+(zq^{2\ell-2})}$$

which upon invoking the root of unity condition $q^{N'}=q^N=1$ gives

$$\sum_{\ell=1}^{N}\frac{q^{-2\ell S^z}(zq^{2\ell}-1)^M}{\mathcal{Q}^+(zq^{2\ell})\mathcal{Q}^+(zq^{2\ell-2})}=0\ . \tag{68}$$

This fact together with the identification $Q^+=\mathcal{Q}^+$ and (56) implies the vanishing of the corresponding eigenvalue of the transfer matrix. This does not mean that the second linear independent solution $\mathcal{Q}^-$ does not exist, it simply states that the normalization constant $\mathcal{N}_{\mu=1}$ in the definition (29) of $Q^-\neq\mathcal{Q}^-$ is zero. Note also that (68) applies to non-degenerate states only, which decrease in number as $M\gg N$ due to the loop algebra symmetry at roots of unity [20].

| $M$ | 3 | 5 | 7 | 9 |
|---|---|---|---|---|
| $N=3$ | 1/3 | 1/10 | 1/35 | 1/126 |
| $N=5$ | 3/3 | 8/10 | 21/35 | 55/126 |
| $N=7$ | 3/3 | 10/10 | 33/35 | 108/126 |

**Table 1**. Shown are the number of "maximal" solutions to the Bethe equations (i.e. $n_+=M/2-S^z$ Bethe roots above and $n_-=M-n_+$ below the equator with $S^z=1/2$) over the dimension of the spin-1/2 sector. The deformation parameter is chosen to be $\mathrm{q}=\exp(2\pi i/N)$. For $N$=3,5 it has been checked that the number of "maximal Bethe states" matches the dimension of the kernel of the auxiliary matrix.

In fact, for $N=3$ in the spin-sector $S^z=\pm1/2$ there is only one state with the expected number of Bethe roots above and below the equator, the groundstate. This is in agreement with the results in [16]. However, our starting point is different from the one in [16]. Instead of making a conjecture on the explicit form of the groundstate of the six-vertex model, we simply start from the assumption that there exists an eigenstate with $m=(M-1)/2$ Bethe roots in the spin $S^z=1/2$ sector. According to (32) complete strings cannot be present and thus (58) must be a constant. But because of (68) with $Q^+=\mathcal{Q}^+$ this constant is vanishing and we have the difference equation,

$$(1-z)^M\mathcal{Q}^+(zq^2)+q^{-1}(1-zq^2)^MQ^+(zq^{-2})+q^{-2}(1-zq^{-2})^MQ^+(z)=0\ . \tag{69}$$

As our conventions differ from Stroganov's we review his calculation in the appendix and show that (69) has a unique solution which can be expressed in terms of hypergeometric functions. The same holds true for the second linear independent solution $\mathcal{Q}^-$ which has $m+1$ roots.

In the case of general $N\in2\mathbb{N}+1$ similar difference equations follow. Let $M=2m+1$, then the eigenvalues of singlet states with $n_\infty=n_S=0$ in the spin $S^z=1/2$ sector with $N\geq3$ satisfy

$$\sum_{\ell=1}^{N}q^{\mp\ell}f_N(zq^{2\ell})=0,\qquad f_N(z)=(1-z)^M\prod_{\ell=1}^{N-2}\mathcal{Q}^\pm(zq^{2\ell})$$

implying the following sum rules in terms of the elementary symmetric polynomials (66) in the $m$ Bethe roots above and the $m+1$ Bethe roots below the equator,

$$0=\sum_{k+l=n}\binom{M}{k}\sum_{k_1+\cdots+k_{N-2}=l}\prod_{j=1}^{N-2}q^{-2j\,k_j}e^{\pm}_{k_j}\;. \tag{70}$$

Here the integer $n$ takes all values in the range

$$0<n=\frac{N\pm1}{2}\bmod N\leq N\;\frac{M\mp1}{2}\pm1 \tag{71}$$

and the different summation variables run over the intervals,

$$0\leq k\leq M,\qquad 0\leq k_j\leq\frac{M\mp1}{2}\;. \tag{72}$$

In general the set of equations (70) is of order $N-2$ and only for $N=3$ becomes linear in the variables $\{e^{\pm}_k\}$, where the equations are particularly simple to solve. Nevertheless, these sum rules are still an advantage over the Bethe ansatz equations which are of order $M$.

# 8 Conclusions

Let us summarize the new results obtained for the six-vertex model at roots of unity. First of all the discussion has been extended from even to odd spin-chains exploiting that the construction procedure for the auxiliary matrices in [7] does not have the same limitations as the one in Baxter's book [21]. This allowed us to reveal the major differences in the spectrum of the six-vertex model at roots of unity between these two cases.

1. When the length of the spin-chain is even there are degeneracies in the spectrum of the transfer matrix for all roots of unity. These degeneracies are reflected in the spectrum of the auxiliary matrices through the occurrence of "complete string factors", see (58) in the text. The number of these strings, i.e. the degree $n_S$ of the polynomial $P_S$ in (27), determines the degeneracy of the corresponding eigenspace of the transfer matrix to be $2^{n_S}$. This is in accordance with the observations made in [20]. In order to arrive at this result we made use of the crucial functional equation (51) which severely restricts the dependence of the complete string factors on the free parameter $\mu$ entering the definition of the auxiliary matrix (19). In addition, we employed (51) to prove the identity (58) which states that the string factors are determined (up to their dependence on the aforementioned parameter $\mu$) by the solution to the Bethe ansatz equations and the number of infinite Bethe roots which fix the eigenvalue of the transfer matrix. These results had previously been proved for $N=3$ only and conjectured to hold true for $N>3$ [8]. Moreover, we deduced that the Bethe roots appear twice in the eigenvalue of the auxiliary matrices, once in the factor $P_B=Q^+$ and once multiplied by the factor $\mu^2$ in the factor $P_\mu$ of the eigenvalue (27). A second linear independent solution to the $TQ$-equation was not found.

2. For spin-chains with an odd number of sites the novel feature was the appearance of such a second linear independent solution to the $TQ$-equation below the equator.

   For primitive roots of unity of odd order this second solution does not exist for all eigenstates of the transfer matrix, but only for those which are singlets and have no infinite roots. For these eigenstates of the transfer matrix we have shown that due to the functional equation (51) the corresponding eigenvalues of the auxiliary matrices must vanish, see equation (68) in the text. The number of these states, i.e. the dimension of the kernel of the auxiliary matrix, will become smaller as the length of the spin-chain starts to exceed the order of the root of unity, i.e. $M \gg N$. (For $N = 3$ we in particular saw that there is only one such singlet state for all odd $M$ and it corresponds to Stroganov's solutions of the $TQ$-equation for the groundstate of the $XXZ$ spin-chain at $\Delta = -1/2$ [16].) This decrease in number can be understood in terms of the loop algebra symmetry [20] of the six-vertex transfer matrix. As the length of the spin-chain $M$ grows more and more of the transfer matrix' eigenstates organize into larger and larger multiplets spanning the irreducible representations. Similar to the case of even spin-chains these degenerate states within the multiplets give rise to complete strings in the eigenvalues of the auxiliary matrices with the same formula $2^{n_S}$ yielding the multiplicity of the transfer matrix eigenvalue.

   For primitive roots of unity of even order the solution below the equator always exists, here, however, the eigenvalues of the transfer matrix do not vanish. We showed the absence of infinite Bethe roots as well as complete strings, leaving at most a double degeneracy in the spectrum of the transfer matrix due to spin-reversal symmetry. The latter is broken by the auxiliary matrices and we used this fact to identify $P_B = Q^+$ and $P_\mu = Q^-$ with the solutions to the $TQ$-equation above and below the equator, respectively. The explicit construction of the $Q$-matrices in [7], see the definition (19) in this article, guarantees therefore the existence of these two solutions, a fact implicitly assumed in [14] for the case of "generic $q$". What is still lacking at the moment is an understanding of the physical significance behind the existence of two linear independent solutions opposed to the case when there is only one. We hope to address this question in a future publication.

In this article we have focussed on the case when $q$ is a root of unity to discuss the spectra of the auxiliary matrices constructed in [7]. But as mentioned in the introduction analogous $Q$-operators have also been constructed when $q$ is not a root of unity [6]. Their spectra together with the resolution of certain convergence problems originating from an infinite-dimensional auxiliary space have been discussed in [9]. As explained therein one in general needs to impose quasi-periodic boundary conditions on the lattice in order to obtain a well-defined auxiliary matrix. For instance, in the critical regime $|q| = 1$ the twist parameter $\lambda$ has to be of modulus smaller than one, $|\lambda| < 1$, to guarantee absolute convergence [9]. This twist parameter enters the definition of the fusion matrices (7) as

$$T^{(n+1)}(zq^{-n-1}) = \operatorname*{Tr}_0 \lambda^{\pi^{(n)}(h_1)\otimes 1} L_{0M}^{(n+1)}(zq^{n+1}) \cdots L_{01}^{(n+1)}(zq^{-n-1}) \tag{73}$$

and modifies the Wronskian (31) in the following manner,

$$\lambda^{-1}q^{S^z}\mathcal{Q}^+(zq^2)\mathcal{Q}^-(z)-\lambda q^{-S^z}\mathcal{Q}^+(z)\mathcal{Q}^-(zq^2)=(\lambda^{-1}q^{S^z}-\lambda q^{-S^z})(1-zq^2)^M\;. \quad (74)$$

Numerical computations show that both solutions $\mathcal{Q}^{\pm}$ exist for $M$ even and odd. Proceeding similar as we did for the case of odd roots of unity we can iteratively solve this equation for, say $\mathcal{Q}^+$, to obtain

$$\mathcal{Q}^-(z)=\lim_{n\to\infty}\lambda^{2n}q^{-2nS^z}\frac{\mathcal{Q}^+(z)\mathcal{Q}^-(zq^{2n})}{\mathcal{Q}^+(zq^{2n})} \\ +(1-\lambda^2q^{-2S^z})\mathcal{Q}^+(z)\sum_{\ell=0}^{\infty}\frac{\lambda^{2\ell}q^{-2\ell S^z}(1-zq^{2\ell+2})^M}{\mathcal{Q}^+(zq^{2\ell+2})\mathcal{Q}^+(zq^{2\ell})}\;.$$

In the limit $n\to\infty$ the first term on the right hand side tends to zero as $|\lambda|<1$ and $|q|=1$. The above expression then matches the results obtained for the eigenvalues of the $Q$-operator from the algebraic Bethe ansatz, see equations (75-78) in [9],

$$Q_{\leq}(z;r_0,r_1)=(-1)^M r_0^{-S^z}(1-\lambda^2q^{-2S^z})Q^+(zr_1)Q^-(z)\;.$$

Here, up to some trivial normalization factors, we can identify $Q^{\pm}\propto\mathcal{Q}^{\pm}$. Thus, similar as in the root-of-unity case the spectrum of the auxiliary matrices [6, 9] when $q$ is not a root of unity contains both solutions to the $TQ$-equation, the one above and the one below the equator. Moreover, these two solutions are related by the analogue of the formula (56) where the summation extends now over an infinite interval.

In the text we commented on a similarity between the relation (56) and a eight-vertex functional equation conjectured by Fabricius and McCoy [4, 5] for Baxter's 1972 auxiliary matrix $Q_{8\text{v}}$ [2] at coupling values $\eta=mK/N'$ (see equation (3.10) in [4] or (3.1) in [5]),

$$e^{-i\pi uM/2K}Q_{8\text{v}}(u-iK')= \\ A\;Q_{8\text{v}}(u)\sum_{\ell=0}^{N'-1}\frac{h^M(u-(2\ell+1)K/N')}{Q_{8\text{v}}(u-2\ell K/N')Q_{8\text{v}}(u-(2\ell+1)K/N')}\;. \quad (75)$$

Here $h(u)=\theta_4(0)\theta_1(\pi u/2K)\theta_4(\pi u/2K)$ in terms of Jacobi's theta-functions with modular parameter $p=\exp(-\pi K'/K)$. We have made the replacements $L\to N'$ (the order of the root of unity) and $N\to M$ (the length of the spin-chain) in the notation of [4, 5]. For general $N'$ the explicit form of the constant $A$ is as yet unknown. This functional equation has been proved for the free fermion case when $M$ is even and numerically verified for coupling values corresponding to roots of unity of order three, see the comment after (3.1) in [5].

If one identifies $Q_{8\text{v}}(u-iK')$ with $Q^-(e^uq^{-1})$ and $Q_{8\text{v}}(u)$ with $Q^+(e^uq^{-1})$ in the six-vertex limit the similarity becomes apparent. This is further supported by the observation that the transformation $u\to u-iK'$ corresponds to spin-reversal in the eight-vertex Boltzmann weights‡. It is tempting to speculate on further identities such

‡The author is thankful to Barry McCoy for discussions on this point.

as the elliptic analogue of the relation (62) for instance. While these identifications can be conjectured and numerically investigated on the level of eigenvalues, the analogous construction of the eight-vertex $Q$-matrices corresponding to (19), which would allow one to prove existence, is a more complicated problem.

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# A Derivation of the functional equation

As mentioned earlier the proof of the functional equation (51) employs representation theory and is deduced from the following non-split exact sequence describing the decomposition of the tensor product $\pi_w^\mu \otimes \pi_1^\nu$,

$$0 \to \pi_{w'}^{\mu'} \overset{\imath}{\hookrightarrow} \pi_w^\mu \otimes \pi_1^\nu \overset{\tau}{\to} \pi_{w''}^{\mu''} \otimes \pi_{z'}^{(N'-2)} \to 0 \ . \tag{76}$$

The various parameters appearing in the representations are not all independent but satisfy the relations

$$w = \mu\nu q^2, \quad \mu' = \mu\nu q, \quad w' = \mu q, \quad \mu'' = \mu\nu q^{-N'+1}, \quad w'' = \mu q^{N'+1}, \quad z' = \nu q^{N'+1} \ . \tag{77}$$

Notice that we have set the second evaluation parameter in the tensor product $\pi_w^\mu \otimes \pi_1^\nu$ equal to one. The general case $\pi_w^\mu \otimes \pi_u^\nu$ is obtained by simply replacing $w \to wu$, $w' \to w'u$ and $w'' \to w''u$, $z' \to z'u$. The line of argument is analogous to the one applied in [7] to derive the $TQ$-equation via (24) and the proof of (51) in [8] for $N = 3$, whence we will be rather brief in presenting the various steps of the proof.

**The inclusion.** First we determine the subrepresentation $\pi_{w'}^{\mu'}$ contained in the tensor product $\pi_w^\mu \otimes \pi_1^\nu$ when $w$ is tuned to the special value given in (52). This will yield the first term on the right hand side of (51). The corresponding inclusion map $\imath : \pi_{w'}^{\mu'} \hookrightarrow \pi_w^\mu \otimes \pi_1^\nu$ is determined by identifying the lowest weight vectors in both representations,

$$\pi_{w'}^{\mu'} \ni |0\rangle \overset{\imath}{\hookrightarrow} |0\rangle \otimes |0\rangle \in \pi_w^\mu \otimes \pi_1^\nu \ . \tag{78}$$

The remaining relations for the rest of the vectors in the included subrepresentation is obtained by successive action of the quantum group generators via the formula

$$\pi_{w'}^{\mu'}(x)\,|0\rangle \overset{\imath}{\hookrightarrow} (\pi_w^\mu \otimes \pi_1^\nu)\Delta(x)\,|0\rangle \otimes |0\rangle \ .$$

The above formula suffices to compute the various parameters. For instance, the parameter $\mu'$ in (77) is computed from the action of the Cartan element $x = q^{h_1}$. The

evaluation parameters $w, w'$ are deduced as follows. First act with $f_1$ on the lowest weight vector to obtain

$$\pi^{\mu'}_{w'}(f_1)\,|0\rangle = |1\rangle \overset{\imath}{\hookrightarrow} (\pi^{\mu}_{w}\otimes\pi^{\nu}_{1})\Delta(f_1)\,|0\rangle\otimes|0\rangle = \nu q\,|1\rangle\otimes|0\rangle + |0\rangle\otimes|1\rangle\ .$$

Alternatively, one obtains via the generator $e_0$,

$$\pi^{\mu'}_{w'}(e_0)\,|0\rangle = w'\,|1\rangle \overset{\imath}{\hookrightarrow} (\pi^{\mu}_{w}\otimes\pi^{\nu}_{1})\Delta(e_0)\,|0\rangle\otimes|0\rangle = w\,|1\rangle\otimes|0\rangle + \mu q\,|0\rangle\otimes|1\rangle\ .$$

Comparing both results leads to the stated values of $w, w'$ in (77). The scalar coefficient in front of the first term on the right hand side of the functional equation (51) is obtained from the identity

$$L^{\mu}_{13}(z\mu q^2)L^{\nu}_{23}(z/\nu)(\imath\otimes 1) = (zq^2-1)(\imath\otimes 1)L^{\mu\nu q}(z\mu q/\nu) \tag{79}$$

which can be easily verified by acting on the lowest weight vector.

**The projection.** In order to compute the second term of the functional equation (51) we need to determine the representation in the quotient space $\pi^{\mu}_{w}\otimes\pi^{\nu}_{1}/\pi^{\mu'}_{w'}$ which is projected out by the map $\tau$. One proceeds in an analogous manner. The projection map $\tau$ is fixed by identifying this time the highest weight vector in both representation spaces,

$$\pi^{\mu}_{w}\otimes\pi^{\nu}_{1} \ni \big|N'-1\big\rangle\otimes\big|N'-1\big\rangle \xrightarrow{\tau} \big|N'-1\big\rangle\otimes\big|N'-2\big\rangle \in \pi^{\mu''}_{w''}\otimes\pi^{(N'-2)}_{z'}\ . \tag{80}$$

Again the value of $\mu''$ can be inferred from the action of the Cartan element $x = q^{h_1}$ via the formula

$$(\pi^{\mu}_{w}\otimes\pi^{\nu}_{1})\Delta(x)\,\big|N'-1\big\rangle\otimes\big|N'-1\big\rangle \xrightarrow{\tau} (\pi^{\mu''}_{w''}\otimes\pi^{(N'-2)}_{z'})\Delta(x)\,\big|N'-1\big\rangle\otimes\big|N'-2\big\rangle\ .$$

Setting $x = f_1 f_0$ and $x = e_0 e_1$ one obtains the desired evaluation parameters $w''$ and $z'$ detailed in (77). For instance, from the left hand side of the above equation one obtains

$$\begin{aligned}&(q-q^{-1})^2(\pi^{\mu}_{w}\otimes\pi^{\nu}_{1})\Delta(f_1f_0)\,\big|N'-1\big\rangle\otimes\big|N'-1\big\rangle = \\ &\qquad\left\{\frac{\mu+\mu^{-1}-\mu q^{-2}-\mu^{-1}q^2}{w} + \nu+\nu^{-1}-\nu q^{-2}-\nu^{-1}q^2\right\}\big|N'-1\big\rangle\otimes\big|N'-1\big\rangle\end{aligned}$$

while the right hand side is computed to

$$\begin{aligned}&(q-q^{-1})^2(\pi^{\mu''}_{w''}\otimes\pi^{(N'-2)}_{z'})\Delta(f_1f_0)\,\big|N'-1\big\rangle\otimes\big|N'-2\big\rangle = \\ &\qquad\frac{\mu\nu q^{-N'+1}+(\mu\nu)^{-1}q^{N'-1}-\mu\nu q^{-N'-1}-(\mu\nu)^{-1}q^{N'+1}}{w''}\,\big|N'-1\big\rangle\otimes\big|N'-2\big\rangle \\ &\qquad\qquad + \frac{(q-q^{-1})^2[N'-2]_q}{z'}\,\big|N'-1\big\rangle\otimes\big|N'-2\big\rangle\ .\end{aligned}$$

Matching the coefficients in both results yields the stated expressions for the parameters. In the case of the quotient projection there is only a trivial additional scalar factor as we have the equality

$$(\tau\otimes 1)L^{\mu}_{13}(z\mu q^2)L^{\nu}_{23}(z/\nu) = q^{N'}L^{\mu\nu q^{-N'+1}}_{13}(z\mu q^{-N'+1}/\nu)L^{(N'-2)}_{23}(zq^{N'+1})(\tau\otimes 1)\ . \tag{81}$$

Again this is most easily calculated by acting with both sides of the equation on the highest weight vector. This completes the proof of the functional equation.

## B Stroganov's solution revisited

As explained in the text the assumption that there exists an eigenstate with $m = (M-1)/2$ Bethe roots in the spin $S^z = 1/2$ sector implies via (32), (58) and (68) with $Q^+ = \mathcal{Q}^+$ that we have the difference equation,

$$(1-z)^M Q^+(zq^2) + q^{-1}(1-zq^2)^M Q^+(zq^{-2}) + q^{-2}(1-zq^{-2})^M Q^+(z) = 0 \ . \tag{82}$$

Expanding

$$(1-z)^M Q^+(zq^2) = 1 + \sum_{n=1}^{3m+1} c_n z^n$$

we infer that the difference equation implies

$$c_n = 0 \quad \text{if} \quad n = 2 \bmod 3 \ .$$

The remaining coefficients can be determined from the fact that $(1-z)^M Q^+(zq^2)$ has an $M$-fold zero at $z = 1$. Applying the method of Lagrange interpolating polynomials, similar as it has been done in [27, 16], one finds the ratios

$$\frac{c_{3n+3}}{c_{3n}} = \frac{(n-m)(n-m-1/3)}{(n+1)(n+2/3)} \qquad \text{and} \qquad \frac{c_{3n+4}}{c_{3n+1}} = \frac{(n-m)(n+1/3-m)}{(n+1)(n+4/3)}$$

together with

$$c_0 = 1 \qquad \text{and} \qquad c_1 = -(4/3)_m/(2/3)_m \ .$$

Here $(x)_m$ is the Pochhammer symbol. From the ratios of the coefficients we infer that there is a unique solution which can be expressed in terms of hypergeometric functions

$$(1-z)^M Q^+(zq^2) = {}_2F_1(-m, -m-\tfrac{1}{3}, \tfrac{2}{3}; z^3) - \frac{(\frac{4}{3})_m}{(\frac{2}{3})_m}\, z\, {}_2F_1(\tfrac{1}{3}-m, -m, \tfrac{4}{3}; z^3) \ . \tag{83}$$

Note that this solution is not simply obtained by multiplying Stroganov's solution (11) in [16] with an exponential factor. This is due to the fact that we solved the difference equation (69) in terms of polynomials which are regular at origin, while Stroganov's solution applies to Laurent series.

**Second solution.** Besides the solution for the Bethe polynomial we just obtained, there is a second solution "beyond the equator" as it possesses $m+1$ roots. Set

$$(1-z)^M \mathcal{Q}^-(zq^2) = 1 + \sum_{n=1}^{3m+2} c'_n z^n$$

then it obeys the difference equation with $S^z = -1/2$, i.e.

$$(1-z)^M \mathcal{Q}^-(zq^2) + q(1-zq^2)^M \mathcal{Q}^-(zq^{-2}) + q^2(1-zq^{-2})^M \mathcal{Q}^-(z) = 0 \ .$$

This implies for the coefficients

$$c'_n = 0 \quad \text{if} \quad n = 1 \bmod 3 \ .$$

As before the solution to this set of equations can be expressed in terms of hypergeometric functions,

$$(1-z)^M \mathcal{Q}^-(zq^2) = {}_2F_1(-m, -m-\tfrac{2}{3}, \tfrac{1}{3}; z^3) - \frac{(\frac{5}{3})_m}{(\frac{1}{3})_m}\, z^2\, {}_2F_1(\tfrac{2}{3}-m, -m, \tfrac{5}{3}; z^3) \ . \tag{84}$$

**Groundstate eigenvalue.** From the difference equation it follows that the eigenvalue of the transfer matrix is given by

$$T(z)P_B = q^{\pm\frac{1}{2}}(z-1)^M \frac{\mathcal{Q}^{\pm}(zq^2)}{\mathcal{Q}^{\pm}(z)} + q^{\mp\frac{1}{2}}(zq^2-1)^M \frac{\mathcal{Q}^{\pm}(zq^{-12})}{\mathcal{Q}^{\pm}(z)} = (zq^{-2}-1)^M$$

which matches the conjecture [17, 18, 19] employed in [16]. The corresponding groundstate eigenvalue of the Hamiltonian is given by $H_{\mathrm{XXZ}} = -M$ .